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Criteria of optimality for sensors' location based on adjoint transformation of observation data interpolation error

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SUMMARY

Criteria of optimality for sensors' location are addressed using an interpolation error transformed by especial adjoint problems. The considered criteria correspond to the analysis error in certain Hessian-based metrics and to the error of some forecast aspect. Both criteria are obtained using adjoint problems that provide computation without the direct use of the Hessian. For a linear inverse heat conduction problem, these criteria are compared and demonstrated promising results when compared with a criterion based on the norm of the interpolation error of observation data. Approaches to sensor set modification using either redistribution of sensors' or refinement of the sensors grid (insertion of additional sensors) are also compared. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: sensor location; targeting; error norms; adjoint methods; sensitivity; sensor set modification

1. INTRODUCTION

Adaptive observations are aimed to improve a forecast by the dynamical retrieval of optimal locations for additional measurements. They employ several different methods that are briefly listed below.

The singular vector approach [1–3] is based on the use of dominant singular vectors (most rapidly growing disturbances) of the integral tangent propagator that may be found from the spectrum of corresponding Fisher information matrix (FIM) or the Hessian [4, 5].

New observations are selected to provide a maximum projection on these singular vectors.

The adjoint sensitivity approach is based on the assumption that a change of the analysis data in zones of maximum gradient of forecast aspect (scalar measure of some forecast quantity of

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interest) causes a maximal change in this aspect, thus additional points of measurements should be placed in these zones [6, 7]. The key element of this technique is the estimation of sensitivity via adjoint equations.

The adjoint targeting strategy by Daescu *et al.* [8, 9] is based on the evaluation of two sensitivity fields: the first associated with the verification cost functional (some forecast score) and the second field associated with the discrepancy functional used in the data assimilation process.

A large number of works in adaptive observations are connected with Kalman filter [10, 11] that provides a natural use of statistical data but requires extensive computer resources.

The combination of adjoint equations (providing fast computations) and the statistical information on measurement and background data error is used in a significant part of publications. In [12] the sensitivity of a forecast aspect to changes in analysis variables was demonstrated to deviate significantly from the sensitivity of the forecast aspect to observations. It is demonstrated to be more advantageous to add measurements in zones of large gradient of the forecast with respect to the observations. The adjoint-derived gradient transformed by Kalman operator is considered as the sensitivity vector that governs optimal sensor locations.

In [13, 14] the optimization of observations was conducted via direct computation of a reduction of the error variance of forecast score caused by additional observations. The variance of forecast score was expressed through adjoint sensitivity gradients and the covariance matrix of analysis (initial state) error. The effect of this matrix changes (caused by a sensors' grid modification) on the forecast score error variance was considered. In this approach, the forecast aspect gradients are estimated in metrics produced by the FIM resulting in a forecast aspect variance. The approach by Bergot and Doerenbecher [13, 14] is very close to the V-optimality condition used in some works on the optimal experiment design [15].

In a significant number of works on the optimum experiment design [16–19], certain measures of the FIM (determinant, maximum or minimum eigenvalues, traces) are used as criteria for an optimal sensor placement. These approaches are rather computationally extensive due to the need to directly operate with FIM or the Hessian.

In [20–22] the estimation of analysis error in dependence on a number of uncertainties (including an observation error) is considered for deterministic and stochastic cases. This estimation is stated using Hessian and the second-order adjoint problem. It is potentially applicable to the adaptive observations. Issues of the control for errors of different origin (including observation and projection errors) are discussed in [23] with the statement of corresponding cost functional and optimality conditions, which may also serve as a guide for adaptive observations.

There exists some analogy (especially significant for the deterministic case) between the search for optimum computational grid and optimum grid of sensors. In some works the minimization of either the local error of approximation [24–26] or the error of some goal functional [27] is used for the retrieval of an optimum computational grid. It is interesting to extend this approach for the search of the optimum sensor locations. However, the direct minimization of the interpolation error of observation data does not provide an account of this error transfer, growth or damping. The natural way to overcome this difficulty is the direct use of an analysis error, caused by the interpolation error. Unfortunately, this approach implies solving an inverse problem (or estimation of the inverse Hessian) at every step of sensors' adaptation iterations thus leading to an extremely high computational burden.

In the present paper some compromise criteria are considered (that are using an adjoint-based transformation of the observation data interpolation error) with the final aim of application to the adaptive observations. These criteria include both the information on an interpolation error and

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the information on this error transfer and do not require solving an inverse problem, thus ensuring their implementation to be computationally inexpensive.

2. PROBLEM STATEMENT

Denote an evolution problem statement as

$$AT_0 = \tilde{T} \tag{1}$$

where A is an operator of the direct problem, A^{-1} is the formal inverse of A, $T(t=0,x)=T_0(x)$ is the initial data ($T_0 \in L_2(Q)$, $Q \subset R^n$, $n=1,2,3$), $\tilde{T}$ is the exact solution at a final time, $T^{\text{obs}}(x_i)$ are the observation data at a final time, $\{x_i\}$ denotes sensors' coordinates ($i=1\dots J_{\text{sen}}$), P is the projection operator from a total field to the sensor positions ($PAT_0 = T^{\text{obs}}$). Herein, all scalar products $(f,g)_{L_2(Q)}$ are for brevity denoted as (f,g), all norms are considered in the following sense: $\|f\|^2 = (f,f)_{L_2(Q)}$ if another meaning is not specified.

The inverse problem of the estimation of the initial data T_0 from the data at the final time may be formally stated as

$$T_0 = A^{-1}\tilde{T} \tag{2}$$

In data assimilation problems instead of $\tilde{T}$ we have observations T^{obs} also made at a final time but on the discrete grid of sensors. The need for some interpolation causes the additional error that can be used as a measure of the quality of sensors' location.

In the present paper the influence of interpolation error is considered, while the impact of measurement error is neglected (measurements are considered to be precise). We address to the search for sensors' locations x_i which are optimal for the estimation of T_0 from two perspectives:

- minimization of the analysis error norm $\|\delta T_0\|$ in special metrics (a statement similar to inverse retrospective problems);
- minimization of the valuable functional error $|\delta J|$ estimated also in special metrics (a statement similar to forecast problems).

In data assimilation problems [1–3, 6–14], usually, the cost functional (discrepancy between observations and model calculations) is minimized and assumes the following form:

$$\varepsilon_1 = (PAT_0 - T^{\text{obs}}, M_1(PAT_0 - T^{\text{obs}})^{\mathrm{T}})/2 \tag{3}$$

where M_1 is a metric tensor for the deterministic approach. In stochastic events it means an inverse covariance matrix ($M_1 = C^{-1}$) of observation error (this implies weighting different sensors in accordance with their precision). The covariance matrix of analysis error may be determined via the inverse Hessian as

$$H^{-1} = (P^* A^* M_1 PA)^{-1} \tag{4}$$

Information regarding fast algorithms for the Hessian calculation may be found in [28, 29].

Herein, we consider another approach. We interpolate the observations $T^{\text{obs}} \to T^{\text{obs}}_{\text{int}}$ on the total computation domain Q

$$T^{\text{obs}}_{\text{int}} = R \cdot T^{\text{obs}} \tag{5}$$

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where R is some interpolation operator (concrete form of R, used herein, is specified in the section of numerical tests), $\delta T_{\text{int}}^{\text{obs}}$ is an interpolation error

$$T_{\text{int}}^{\text{obs}} = \tilde{T} + \delta T_{\text{int}}^{\text{obs}} \tag{6}$$

The estimation of T_0 (solving an inverse problem $T_0 = A^{-1} T_{\text{int}}^{\text{obs}}$) may be done in the optimization statement by a minimization of the discrepancy of the forecast and the interpolation of observations

$$\varepsilon = (AT_0 - T_{\text{int}}^{\text{obs}}, M(AT_0 - T_{\text{int}}^{\text{obs}}))/2 \tag{7}$$

The gradient of the discrepancy assumes the following form:

$$\nabla\varepsilon = A^* M(AT_0 - T_{\text{int}}^{\text{obs}}) \tag{8}$$

where A^* is the adjoint problem operator. The corresponding form of FIM A^*MA is a symmetrical positive-definite matrix that may be used as a metric tensor. In the vicinity of the optimal solution it is equal to the Hessian of discrepancy that is denoted here as H_1.

In the present paper an optimal location of sensors is considered from the viewpoint of the minimization of certain easily computable norm of the analysis error δT_0. As raw information we use estimations of the interpolation error of observation data $\delta T_{\text{int}}^{\text{obs}}$ (that may assume different forms as surveyed in [30], for example). Corresponding error of T_0 may be expressed as $\delta T_0 = A^{-1} \delta T_{\text{int}}^{\text{obs}}$, or, accounting

$$\delta\nabla\varepsilon = -A^* M \delta T_{\text{int}}^{\text{obs}} \tag{9}$$

as

$$\delta T_0 = -H_1^{-1} \delta\nabla\varepsilon \tag{10}$$

We denote $\nabla\varepsilon(T_0)$ as $\delta\nabla\varepsilon$, herein since it has some specific features (for example, even at the exact solution $\nabla\varepsilon(T_0) \neq 0$ due to the presence of $\delta T_{\text{int}}^{\text{obs}}$).

Consider several norms of δT_0 from a standpoint of their computational convenience and physical meanings.

A norm of interpolation error

$$\|\delta T_{\text{int}}^{\text{obs}}\| = (\delta T_{\text{int}}^{\text{obs}}, \delta T_{\text{int}}^{\text{obs}})^{1/2} \tag{11}$$

may be used for the search of the optimal sensor location similarly to the methods of computation of grid adaptation [24]. $\|\delta T_{\text{int}}^{\text{obs}}\|$ corresponds to the norm of δT_0 in a metric engendered by the matrix $H_E = A^* A$:

$$(\delta T_{\text{int}}^{\text{obs}}, \delta T_{\text{int}}^{\text{obs}}) = (\delta T_0, H_E \delta T_0) = \|\delta T_0\|_{H_E}^2 \tag{12}$$

The expression $(\delta T_{\text{int}}^{\text{obs}}, \delta T_{\text{int}}^{\text{obs}})$ may be directly calculated from the observations without solving the main problem and, thus, it does not contain information about the considered problem which constitutes its obvious shortcoming.

The norm of error of the inverse problem solution (analysis error)

$$\begin{aligned} \|\delta T_0\|^2 &= (\delta T_0, \delta T_0) = (A^{-1}\delta T_{\text{int}}^{\text{obs}}, A^{-1}\delta T_{\text{int}}^{\text{obs}}) = (\delta T_{\text{int}}^{\text{obs}}, A^{-1*}A^{-1}\delta T_{\text{int}}^{\text{obs}}) \\ &= (\delta T_{\text{int}}^{\text{obs}}, H_E^{-1}\delta T_{\text{int}}^{\text{obs}}) \end{aligned} \tag{13}$$

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may be considered as a 'natural' criterion for the optimal location of sensors. The inverse Hessian serves herein as a metrics tensor in the space of measurements. The obvious disadvantage of this approach is caused by the high computational burden of H_E^{-1} calculation, especially in an iterative process of sensors' allocation, and instabilities occurring when inverting the Hessian.

In the present paper we suggest to use the norm of the gradient variation caused by $\delta T_{\text{int}}^{\text{obs}}$

$$\|\delta\nabla\varepsilon\|^2 = (\delta\nabla\varepsilon, \delta\nabla\varepsilon) \tag{14}$$

which may be computed using direct and especial adjoint (loaded by the interpolation error) problems without the direct use of the Hessian. Taking into account (9), we obtain

$$\begin{aligned}(\delta\nabla\varepsilon, \delta\nabla\varepsilon) &= (A^* M\delta T_{\text{int}}^{\text{obs}}, A^* M\delta T_{\text{int}}^{\text{obs}}) = (\delta T_{\text{int}}^{\text{obs}}, M^* AA^* M\delta T_{\text{int}}^{\text{obs}}) = (A\delta T_0, M^* AA^* MA\delta T_0)\\ &= (\delta T_0, A^* M^* AA^* MA\delta T_0) = (\delta T_0, H_1^* H_1 \delta T_0) = \|\delta T_0\|^2_{H_1^* H_1}\end{aligned} \tag{15}$$

Thus, the norm of gradient variation is equal to the norm of analysis error in certain Hessian-based metric engendered by matrix $H_1 H_1^*$. This norm is of interest due to its relative simplicity of calculation.

In a set of problems we may be interested in the precise calculation of some valuable functional (forecast aspect), herein denoted as $J(T_0)$. On the solution of an additional adjoint problem ψ connected with this functional (the detailed statement is presented in following section (Equations (35)–(39))) we may determine the variation of the forecast aspect as

$$\delta J = (\psi, \delta T_0) = (\psi, A^{-1}\delta T_{\text{int}}^{\text{obs}}) \tag{16}$$

The estimation of $\delta J = (\psi, \delta T_0)$ implies solving an inverse problem for the estimation of δT_0, and thus it involves a significant computational cost. Thus, in the present paper we consider an alternative expression

$$(\psi, \delta\nabla\varepsilon) = (\psi, A^* M\delta T_{\text{int}}^{\text{obs}}) = (\psi, A^* MA\delta T_0) = (\psi, H_1\delta T_0) = \delta J_{H_1} \tag{17}$$

which can be calculated from two adjoint problems, linked with the forecast aspect and the interpolation error and providing values of ψ and $\delta\nabla\varepsilon$. We treat this value as the variation of the forecast aspect calculated in the special metric.

The values $\|\delta\nabla\varepsilon\|$ and $|(\psi, \delta\nabla\varepsilon)|$ are considered in this paper as criteria for sensors' location optimality and compared with the directly computed $\|\delta T_{\text{int}}^{\text{obs}}\|$. The corresponding value of $\|\delta T_0\|$ is estimated by solving the inverse problem (as the difference of *a priori* known T_0 and the result of inverse problem solving) and used as a reference. In numerical tests we solved the inverse problem by gradient-based iterations (conjugate gradients) that provide an implicit regularization [17].

The optimization of location of sensors in numerical tests was performed by both redistribution and refinement. The redistribution of sensors was conducted by an optimization using conjugate gradients and the simplex method [31]. The refinement means the placement of additional sensors in zones of the large error density. Generally speaking, the redistribution has a greater potential from the viewpoint of optimal sensor configuration and is close to an optimum experiment design [16]. However, it is connected with a great number of algorithmic problems and require high computational resources. In this approach, the grid of sensors tends to be highly irregular. For two dimensional or three dimensional events, an interpolation on the irregular grid (and corresponding error estimation) may cause serious problems [30, 32] and redistribution seems to be very difficult

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or even impossible. A refinement is much simpler from this viewpoint and close to the current practice of adaptive observations.

3. TEST PROBLEM

Let us consider a one-dimensional problem for the determination of an initial temperature distribution $T_0(x)$ from observations at the final moment $T^{\text{obs}}(x_i); i=1\dots J_{\text{sen}}$. In inverse problems, usually, the cost functional having *a summation over the sensors* should be minimized

$$\varepsilon_1(T_0(\cdot))=\sum_{i,j}^{J_{\text{sen}}}(T(t_f,x_i)-T^{\text{obs}}(x_i))M_{1,ij}(T(t_f,x_j)-T^{\text{obs}}(x_j)) \tag{18}$$

In the present paper, we consider another functional with *a summation over all grid nodes*

$$\varepsilon(T_0(\cdot))=\sum_{i,j}^{N}(T(t_f,x_i)-RT^{\text{obs}}(x_i))M_{ij}(T(t_f,x_j)-RT^{\text{obs}}(x_j)) \tag{19}$$

For deriving the adjoint equation the following continuous analog is used:

$$\varepsilon(T_0(\cdot))=\int(T(t_f,x)-T^{\text{obs}}_{\text{int}}(x))M(x,y)(T(t_f,y)-T^{\text{obs}}_{\text{int}}(y))\,\mathrm{d}x\,\mathrm{d}y \tag{20}$$

The direct problem is described by the unsteady one-dimensional heat transfer equation

$$\frac{\partial T}{\partial t}-\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)=0,\quad (t,X)\in\Omega=(0<t<t_f;0<x<X) \tag{21}$$

with boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_0=0,\quad \left.\frac{\partial T}{\partial x}\right|_X=0 \tag{22}$$

The final condition is obtained by an interpolation of observation data

$$T|_{t=t_f}=T^{\text{obs}}_{\text{int}}(x) \tag{23}$$

The error of interpolation may be presented as $\delta T^{\text{obs}}_{\text{int}}(x)=T^{\text{obs}}_{\text{int}}(x)-\tilde{T}(t_f,x)$, where tildes mean the exact solution (in tests $\delta T^{\text{obs}}_{\text{int}}(x)$ is approximately estimated via higher-order terms of the interpolation polynomial).

The initial conditions

$$T|_{t=0}=T_0(x) \tag{24}$$

are unknown and should be estimated. As a result we obtain a retrospective statement of the inverse heat transfer problem [17]. Certainly, the error of $T_0(x)$ grows as $t_f\to\infty$, and there exist restrictions on an admissible interval of observation t_f which may be determined using *a priori* information on $T_0(x)$ in a standard application for inverse problems [17] .

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We may use gradient-based iterative methods to determine $T_0(x)$. The following adjoint problem:

$$\frac{\partial \Psi}{\partial t}+\lambda\frac{\partial^2 \Psi}{\partial x^2}=0 \tag{25}$$

with the boundary condition

$$\left.\frac{\partial \Psi}{\partial x}\right|_0=0, \quad \left.\frac{\partial \Psi}{\partial x}\right|_X=0 \tag{26}$$

and the final condition

$$\Psi(t_f,x)=-\int M(x,y)(T(t_f,y)-T_{\text{int}}^{\text{obs}}(y))\,\mathrm{d}y \tag{27}$$

may be posed. When $M(x,y)=E(x,x)$ ($E(x,x)$ is a continuous analog of the unity matrix) the commonly used form (which is applied in further considerations) is recovered

$$\Psi(t_f,x)=-(T(t_f,x)-T_{\text{int}}^{\text{obs}}(x)) \tag{28}$$

The gradient of the goal functional has a form

$$\nabla\varepsilon(T_0)=-\Psi(0,x) \tag{29}$$

which may be used in the gradient-based optimization procedures.

Let us consider an optimization of sensor locations in the framework of above-mentioned problem.

An optimal placement of sensors based on the criterion $\|\delta\nabla\varepsilon\|^2$

$$\{x_i\}=\arg\min\|\delta\nabla\varepsilon\|^2 \tag{30}$$

implies solving of the following adjoint problem:

$$\frac{\partial \Psi_1}{\partial t}+\lambda\frac{\partial^2 \Psi_1}{\partial x^2}=0 \tag{31}$$

with the boundary condition

$$\left.\frac{\partial \Psi_1}{\partial x}\right|_0=0, \quad \left.\frac{\partial \Psi_1}{\partial x}\right|_X=0 \tag{32}$$

and with the final condition

$$\Psi_1(t_f,x)=\delta T_{\text{int}}^{\text{obs}}(x) \tag{33}$$

As a result, this problem solution provides the value

$$\delta\nabla\varepsilon=-\Psi_1(0,x)$$

whose norm should be minimized.

The search of an optimal sensor location from the criterion $|(\psi,\delta\nabla\varepsilon)|$

$$\{x_i\}=\arg\min|(\psi,\delta\nabla\varepsilon)| \tag{34}$$

includes solving of the additional adjoint problem associated with the goal functional (forecast aspect). In numerical tests, we consider the temperature at some point $T_{\text{est}} = T(t_{\text{est}}, x_{\text{est}})$ as a goal functional

$$J = \int\int_{\Omega} T(t,x)\delta(t - t_{\text{est}})\delta(x - x_{\text{est}})\,\mathrm{d}t\,\mathrm{d}x \tag{35}$$

The corresponding adjoint problem follows:

$$\frac{\partial\psi}{\partial t} + \lambda\frac{\partial^2\psi}{\partial x^2} = 0, \quad (t,x)\in\Omega \tag{36}$$

Boundary condition as:

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=X} = 0, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = 0 \tag{37}$$

We use $t_{\text{est}} = t_f$, then the final condition:

$$\psi(t_f, x) = -\delta(x - x_{\text{est}}) \tag{38}$$

The goal functional variation caused by the analysis error assumes the form

$$\delta J = (\psi(0,x), \delta T_0) \tag{39}$$

Instead of this value we will consider $\delta J_{H_1} = (\psi(0,x), H_1\delta T_0) = (\psi(0,x), \delta\nabla\varepsilon)$ using solutions of (36)–(38) and (31)–(33) which are less computationally expensive.

The direct problems (21)–(24) and adjoint problems (25)–(29), (31)–(33) and (36)–(38) were solved in numerical tests. A finite difference approximation of both the heat transfer equation and the corresponding adjoint equations having a second-order accuracy over time and space was used. An implicit method (implemented by using the Thomas algorithm) was applied for solving these

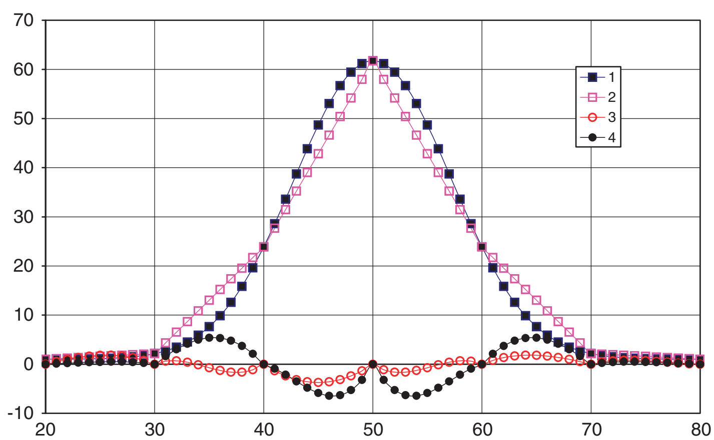

Figure 1. 1—true distribution of measured temperature, 2—interpolated distribution, 3—error estimation (41), 4—true error of interpolation.

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problems. Thermal diffusivity value was taken as $\lambda = 2\times 10^{-7}\,\mathrm{m}^2/\mathrm{s}$. The spatial grid consisted of 50–500 nodes; the temporal integration contained 100–10 000 time steps. The illustrations, presented herein, have been carried out with 100 spatial nodes and 100 time steps. The initial temperature distribution is presented in Figure 2, while the final distribution may be found in Figure 1.

4. NUMERICAL TESTS

A number of numerical tests were performed for the study and comparison of the considered criteria. At a first step (for known $T_0(x)$) the exact magnitude of J is computed, and the inverse retrospective problem is solved for total input data (posed at the every node of computational grid at the final time). The error of an inverse problem solution $(\|\delta T_0\|^2)_1 = \sum_k^N (T_{k,0} - T_{k,0}^{\text{exact}})^2/N$ is determined by comparing it with the known $T_0(x)$ and assumed to be the lowest bound of error.

At a second step the number of sensors and their initial positions $\{x_i^0\}$ are set. For these sensors' data we also solved the inverse problem and obtained an error norm $(\|\delta T_0\|^2)_2$ assumed to be the starting value of the error that should be minimized. The optimization of location of sensors $\{x_i^{\text{new}}\}$ for $(\|\delta T_0\|^2)_2$ reduction is conducted by the minimization of the following functionals of the interpolation error:

1. $\|\delta T_{\text{int}}^{\text{obs}}\|$
2. $\|\delta\nabla\varepsilon\|$
3. $|(\psi, \delta\nabla\varepsilon)|$

or by inserting additional sensors into zones of great density of error. For optimized locations $\{x_i^{\text{new}}\}$ we calculated the observation data, resolved the inverse problem and found $(\|\delta T_0\|^2)_3$ which is compared with the minimal and initial values.

4.1. Estimation of the interpolation error of observation data

Herein, we consider the estimation of $\|\delta T_{\text{int}}^{\text{obs}}\|$ in one-dimensional case. At the initialization stage we have a set of sensors $\{x_i\}, i = 1\ldots J_{\text{sen}}$ and corresponding observations $T^{\text{obs}}(x_i)$. The observations are assumed to be precise since we are studying, herein, the impact of interpolation error. Using these data one may compute an interpolation of observations (at instant t_f) by the Newton interpolation polynomial.

The piecewise constant derivatives are computed for an interpolation error estimation

$$\frac{\partial^2 T^{\text{apr}}}{\partial x_{k+1}^2} = \frac{1}{x_{k+2}-x_k}\left(\frac{T^{\text{obs}}(x_{k+2}) - T^{\text{obs}}(x_{k+1})}{x_{k+2}-x_{k+1}} - \frac{T^{\text{obs}}(x_{k+1}) - T^{\text{obs}}(x_k)}{x_{k+1}-x_k}\right) \tag{40}$$

For linear interpolation, the second term of Newton polynomial (coinciding with Taylor expansion in one dimension) provides the estimation of the interpolation error on (x_k, x_{k+1}) as follows:

$$\delta T_{\text{int}}^{\text{obs}}(x_k, x_{k+1}) = \frac{(x-x_{k+1})(x-x_k)}{2}\frac{\partial^2 T^{\text{apr}}}{\partial x^2} \tag{41}$$

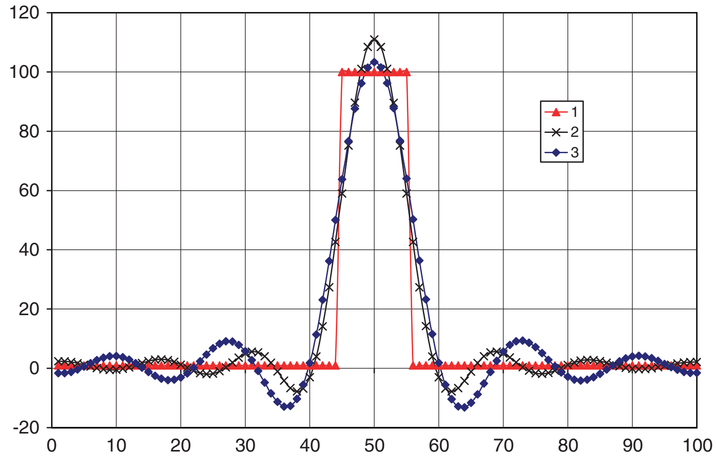

Figure 2. 1—exact T_0, 2—estimation using measurements on all computation grids, 3—estimation using 11 sensors.

True and interpolated distributions of the final (observed) temperature along with the true interpolation error and estimation of this error (41) are presented in Figure 1 as a function of the node number.

After optimization, we find a new set of sensors $\{x_i^{\text{new}}\}, i=1\ldots J_{\text{sen}}$ such that the error

$$\varepsilon(x_i)=\sum_{k=1}^{J_{\text{sen}}-1}\int_{x_k}^{x_{k+1}}(\delta T_{\text{int}}^{\text{obs}})^2\,\mathrm{d}x=\sum_{k=1}^{J_{\text{sen}}-1}\int_{x_k}^{x_{k+1}}\left((x-x_k)(x-x_{k+1})\frac{1}{2}\frac{\partial^2 T^{\text{apr}}}{\partial x^2}\right)^2\mathrm{d}x \tag{42}$$

diminishes. We employed in (42) 'frozen' derivatives (computed from the initial sensors' location and kept constant over iterations). Otherwise, the sensors tend to aggregate in zones of low second derivatives away from the regions containing useful information (zones that provide the trivial solution of minimization problem). This is one of the multiple obstacles in the way of effective sensors' redistributions.

The initial temperature distribution and results of its estimation using different grids of sensors are presented in Figure 2 (11 sensors are located at points $1, 10, 20, \ldots, 100$).

Figures 3 and 4 contain the results of the redistribution via an optimization (using conjugate gradients). The analysis of these data demonstrates the decrease of exact error norm by a factor 3.1 (error distributions are provided in Figure 4) when error estimation (42) dropped by a factor of 2.5 (Figure 3). Thus, reduction of the error estimation $\delta T_{\text{int}}^{\text{obs}}$ leads to a decrease in the true error δT_0.

This approach (sensor redistribution) is difficult from the algorithmic viewpoint and entails a large computational burden caused by the multiple direct problem solution and the need to use new measurements for all sensors.

The refinement (inserting additional sensors into zones of the high error density) is much simpler from the algorithmic viewpoint and it does not require significant additional computations. Results of the refinement (addition of measurements in points 45 and 55 reduces the error by 20%) are presented in Figure 5.

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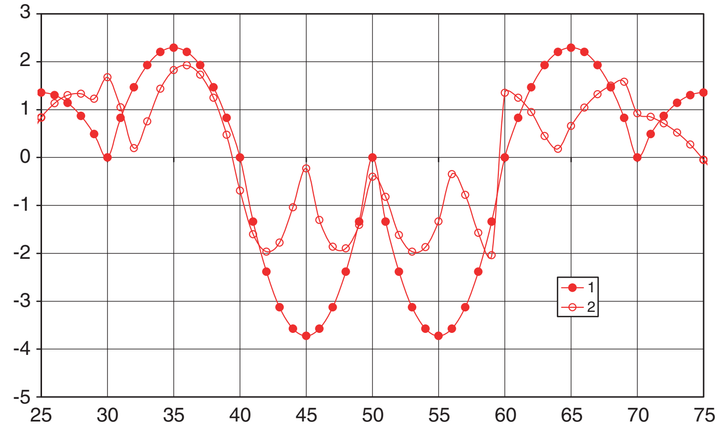

Figure 3. 1—error estimation on initial grid of sensors, 2—error estimation on optimized grid of sensors.

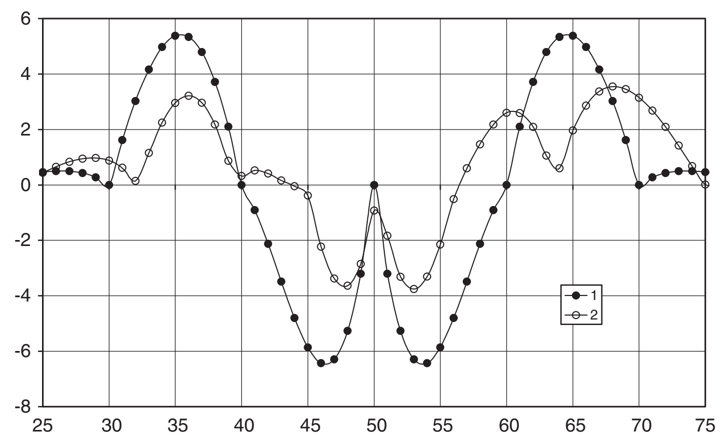

Figure 4. 1—true error on initial grid of sensors, 2—true error on optimized grid of sensors.

Thus, the numerical tests (Figures 3–5) confirm that the local error of interpolation $\delta T_{\mathrm{int}}^{\mathrm{obs}}$ may be estimated and reduced by either a redistribution or a refinement of sensors' grid in such a way that the true error δT_0 also diminishes.

4.2. *Optimization of sensor location via* $\|\delta\nabla\varepsilon\|$ *and* $|(\psi,\delta\nabla\varepsilon)|$

An optimization of sensors' location using $\delta T_{\mathrm{int}}^{\mathrm{obs}}$ is demonstrated in the above section to be feasible but it may be not the best option, because it does not account for features of both the used model and the goal functional. Thus, we consider another criteria $\|\delta\nabla\varepsilon\|$ and $|(\psi,\delta\nabla\varepsilon)|$. Figure 6 enables us to compare the error estimation $\delta T_{\mathrm{int}}^{\mathrm{obs}}$ (41), the true error and $\delta\nabla\varepsilon$.

The density of $\|\delta\nabla\varepsilon\|$ is much smoother when compared with either true error or the estimation error.

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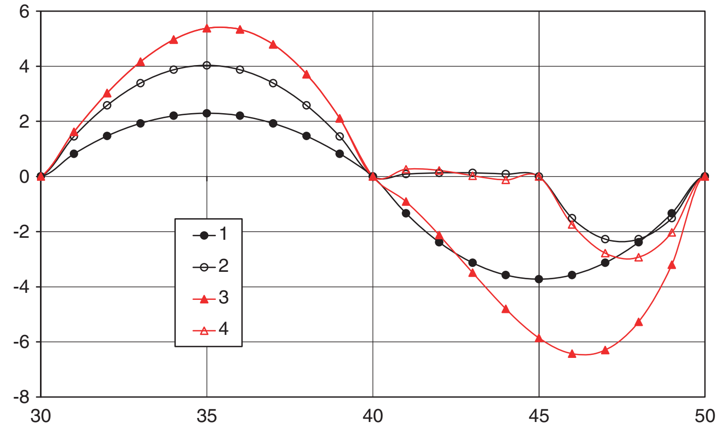

Figure 5. 1—error estimation on initial grid of sensors, 2—error estimation on refined grid of sensors, 3—true error on initial grid, 4—true error on refined grid. (Curves 3 and 4 coincide in zones of unperturbed sensors.)

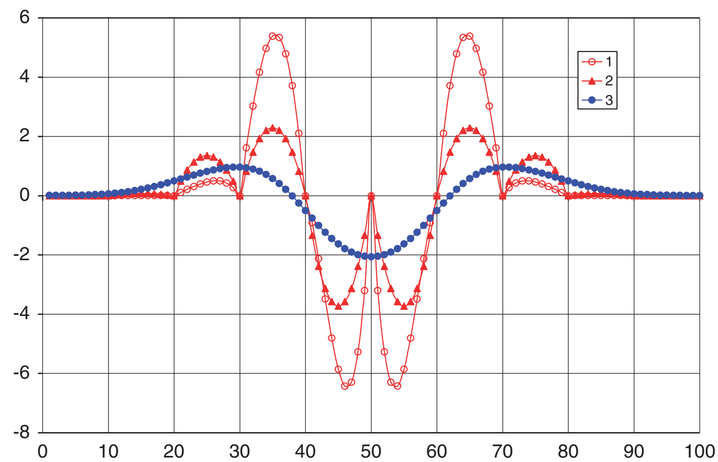

Figure 6. Comparison of error densities. 1—true error, 2—error estimation (42), 3—$\delta\nabla\varepsilon$.

The density error has a significantly different shape if we care for the accuracy of the goal functional. Figure 7 presents distributions of the error density components for the pointwise functional (35).

Initial coordinates of sensors and coordinates obtained by optimization (conjugate gradients) using the above considered criteria are presented in Table I. The quality of optimization (when compared with the initial sensor grid) from the viewpoint of $\|\delta T_0\|^2$ and $\delta J/J$ may be estimated from Table II.

The simplex method [31] used for a minimization of $\|\delta\nabla\varepsilon\|$ and $\|\delta T_{\mathrm{int}}^{\mathrm{obs}}\|$ tends to change the order of sensors and was considered as unacceptable but was improved by adding a special

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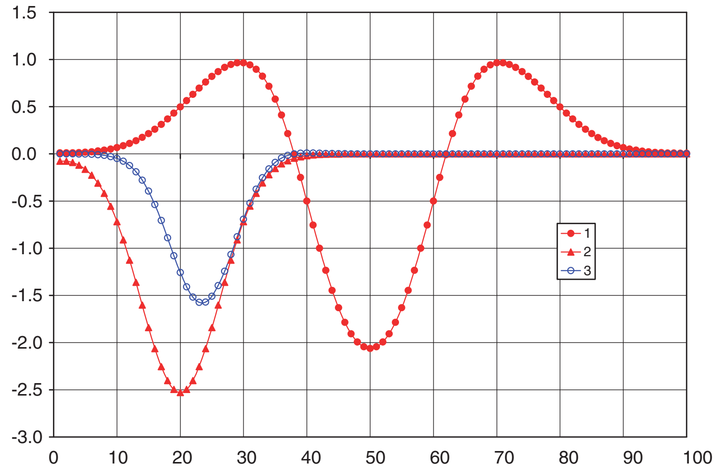

Figure 7. Distributions of density error components for the goal functional. 1—$\delta\nabla\varepsilon$, 2—ψ, 3—$\psi\delta\nabla\varepsilon$.

Table I. Initial and optimized coordinates of sensors.

	Sensor coordinates										
Initial $\{x_i^0\}$	1	4	14	24	34	44	54	64	74	84	100
$\{x_i^{new}\}$, regarding $\|\delta T_{int}^{obs}\|$	1	10.3	18.7	27.7	35.5	45.5	52	62	70	79.5	100
$\{x_i^{new}\}$, regarding $\|\delta\nabla\varepsilon\|^2$	1	18	25.3	32	45	49	53	59	67.5	73.5	100
$\{x_i^{new}\}$, regarding $\|(\psi,\delta\nabla\varepsilon)\|$	1	4	9	14	20	24	30	34	54	84	100

Table II. Analysis error norm and relative goal error for different grid of sensors.

	Total grid data	Initial sensor grid $\{x_i^0\}$	$\{x_i^{new}\}$, optimal regarding $\|\delta T_{int}^{obs}\|$	$\{x_i^{new}\}$, optimal regarding $\|\delta\nabla\varepsilon\|$	$\{x_i^{new}\}$, optimal regarding $\|(\psi\delta\nabla\varepsilon)\|$
$\|\delta T_0\|^2$	1.16	2.53	1.86	1.46	2.56
$\|\delta J\|/J$	0	0.713	0.138	0.067	0.045

penalty term. However, results obtained exhibit a similar quality for both the simplex method and the conjugate gradients. For $|(\psi,\delta\nabla\varepsilon)|$ neither the simplex method nor the conjugate gradients provided acceptable results and a refinement was used for the sensor grid modification.

Data presented in Table II demonstrate the optimization over $\|\delta\nabla\varepsilon\|$ to provide the best results from viewpoint of $\|\delta T_0\|$ when compared with the optimization over$\|\delta T_{int}^{obs}\|$. Naturally, the optimization over $|(\psi,\delta\nabla\varepsilon)|$ is more advantageous from a $\delta J/J$ viewpoint (however, it deteriorates $\|\delta T_0\|^2$).

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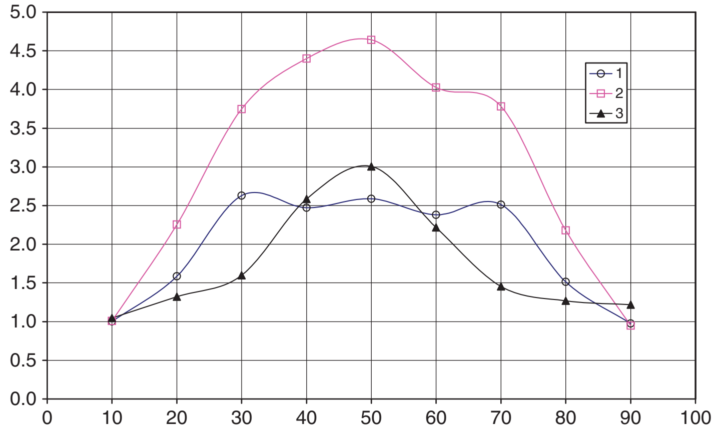

Figure 8. Relative importance of sensors from the viewpoint of different criteria. 1—$\|\delta T_{\rm int}^{\rm obs}\|$, 2—$\|\delta\nabla\varepsilon\|$, 3—$\|\delta T_0\|$.

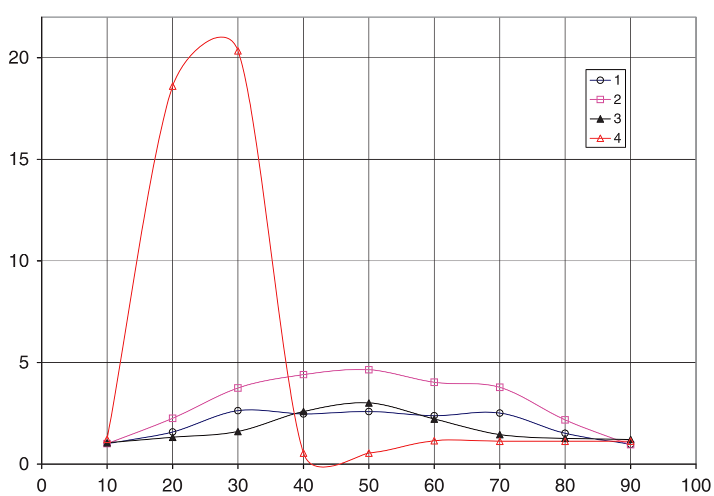

Figure 9. Relative importance of sensors from the viewpoint of different criteria. 1—$\|\delta T_{\rm int}^{\rm obs}\|$, 2—$\|\delta\nabla\varepsilon\|$, 3—$\|\delta T_0\|$, 4—$|(\psi,\delta\nabla\varepsilon)|$.

4.3. *Comparison of impact of different sensors*

If we delete a sensor, this action affects all the above-mentioned measures of errors. Thus, the information provided by a given sensor may be estimated via the increase in the error caused by its deletion.

The results are provided in Figures 8 and 9 as a function of the sensor location. The error is divided by the error corresponding to the intact sensor grid $\|\delta T_{\rm int}^{\rm obs}{}_{{\rm all}-i}\|/\|\delta T_{\rm int}^{\rm obs}{}_{\rm all}\|$, $\|\delta\nabla\varepsilon_{{\rm all}-i}\|/\|\delta\nabla\varepsilon_{\rm all}\|$ and $\|\delta T_{0,{\rm all}-i}\|/\|\delta T_{0,{\rm all}}\|$.

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It may be noted that the relative importance of sensors estimated via $\|\delta\nabla\varepsilon\|$ is closer to the one obtained from $\|\delta T_0\|$, than to the one, obtained from $\|\delta T_{\text{int}}^{\text{obs}}\|$. This argues in favor of using $\|\delta\nabla\varepsilon\|$ for the sensor location optimization. From the viewpoint of goal functional (35), the importance of sensors assumes another form and is presented in Figure 9.

Thus, the approach presented above provides a feasibility for the qualitative evaluation (and comparison) of the information obtained by a selected sensor. The estimation of the relative importance of sensors provides another tool for the sensor grid improvement.

The considered sensor location criteria are presented for a linear model; however, this does not detract from the generality of approach. Both for a linear and for nonlinear statements we should recalculate the sensors' location in dependence on the observed field. Relatively good results may be obtained for a small observation error when the linear approximation of the influence propagation (adjoint problem) is valid.

5. CONCLUSION

The criterion of sensor location optimality $(\delta\nabla\varepsilon, \delta\nabla\varepsilon)$ having a meaning of an analysis error in a special Hessian-based metric and the criterion connected with the goal functional (forecast aspect) error $|(\psi, \delta\nabla\varepsilon)|$ were considered in this research. Both criteria use adjoint problems that transform the interpolation error of observation data and provide for a relatively efficient calculation excluding inverse problem solution or direct use of the Hessian. The numerical tests carried out demonstrated the applicability of these criteria for the adaptive observations and promising results when compared with a criterion based on the norm of the interpolation error.

The optimization-based redistribution of sensors requires significant computational efforts and encounters serious algorithmic problems caused, for example, by possible sensors merging or a change in the sensors' order. The refinement using both criteria $(\delta\nabla\varepsilon, \delta\nabla\varepsilon)$ and $|(\psi, \delta\nabla\varepsilon)|$ provides a faster and more robust improvement of a sensor grid in contrast to the redistribution.

The information provided by a selected sensor may be qualitatively estimated by the increase of some measure of error (provided by the above-mentioned criteria) caused by its deletion. This information depends both on the sensor location in a measured field and on the configuration of the other sensors.

The proposed sensor location criteria were presented and implemented in a linear framework. Extension to nonlinear case will constitute our follow-up research effort.

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